

Absorption of Electro-magnetic Waves in a Magnetized Medium

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In continuation to our earlier work, in which the structure of the vacuum polarisation tensor in a medium was analysed in presence of a background electro-magnetic field, we discuss the absorptive part of the vacuum polarization tensor. Using the real time formalism of finite temperature field theory we calculate the absorptive part of 1-loop vacuum polarisation tensor in the weak field limit ($eB < m^2$). Estimates of the absorption probability are also made for different physical conditions of the background medium.

I. INTRODUCTION

Processes in a magnetised plasma are of interest in widely different physical systems ranging from laboratory plasma to astrophysical objects [1]. Therefore, the propagation of electro-magnetic waves, in a magnetised plasma, continues to evoke much interest. Fortunately, almost all of the terrestrial or the astrophysical systems, barring the newly discovered ‘magnetars’ [2], have magnetic fields smaller than the QED limit ($eB < m_e^2$ i.e., $B \leq 10^{13}$ Gauss). This allows for a weak-field treatment of the plasma processes relevant to such physical situations. Moreover, this treatment is also valid for compact astrophysical objects (viz. white dwarfs or neutron stars) for which the Landau level spacings are quite small compared to the electron Fermi energy [3]. This ensures that the magnetic field does not introduce any spatial anisotropy in the collective plasma behaviour.

In view of this, in an earlier paper [4] (paper-I henceforth) we analysed the structure of the vacuum polarization tensor (denoted by $\Pi_{\mu\nu}$) in a background medium in presence of a uniform external magnetic field, in conformity with Lorenz and gauge invariance. In the weak-field limit we retained terms up-to $\mathcal{O}(B)$ to obtain the field dependence of the vacuum polarisation, calculated at the 1-loop level. As expected, we recovered Faraday rotation, i.e, the phenomenon of the rotation of the plane of polarisation of an electro-magnetic wave passing through an ionised medium in presence of an external magnetic field, from the $\mathcal{O}(B)$ term.

The present paper can be thought of as a sequel to paper-I, in which, we calculated the dispersive part of $\Pi_{\mu\nu}$. In this work we calculate the absorptive part thereby obtaining a complete expression for the polarisation at the 1-loop level. Once again we retain terms up-to $\mathcal{O}(B)$ to investigate the field dependence of the absorptive part which can be further used to evaluate the damping/instability of the photons propagating in a plasma. It is worth noting that the damping/instability probability of a propagating photon due to pair creation/annihilation in a medium can also be obtained from the tree-level dia-

gram using the exact wave-functions of the fermions in an external magnetic field. However, these wave-functions are not well defined in the weak-field limit and therefore use of tree-level amplitudes may not be very accurate. On the other hand the general structure of $\Pi_{\mu\nu}$ has a well defined limit as $eB \rightarrow 0$. Hence this approach is free from the problems of the weak-field limit.

The organization of the document is as follows. In section-II we discuss the basic formalism. Section-III contains the details of the calculation of the absorptive part of the polarisation tensor in presence of a background magnetic field. In section-IV we outline the relation between the polarisation tensor and the dispersion relation. Finally in section-V we discuss our results to a few limiting cases. The appendix contains a few details.

II. FORMALISM

Recall that in presence of an external field, the interaction part of the Lagrangian is given by:

$$\mathcal{L} = - \int j^\mu(x) \mathcal{A}_\mu(x) d^4x, \quad (1)$$

where $j^\mu(x)$ can be defined in terms of the fermion field $\Psi(x)$ (solutions of the equation of motion in presence of an external field) as:

$$j^\mu(x) = -e \bar{\Psi}(x) \gamma^\mu \Psi(x). \quad (2)$$

Correspondingly, the S -matrix for the theory is defined to be:

$$S = T e^{i \int d^4x j^\mu(x) \mathcal{A}_\mu(x)}, \quad (3)$$

where T refers to the time-ordering of the product. Further, the S -matrix can be written in the following form:

$$S = 1 + \sum_{n \geq 1} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n T(\mathcal{L}(x_1) \dots \mathcal{L}(x_n)). \quad (4)$$

It is worth noting that the second term in the right hand side is the usual \mathcal{T} -matrix. Since we are interested

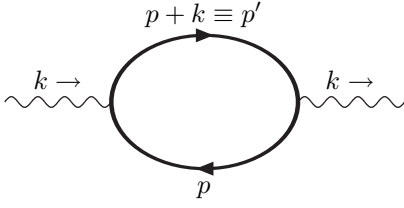


FIG. 1. One-loop diagram for the vacuum polarization.

in fermionic pair-creation/annihilation in presence of a magnetic field in a medium, terms up to $O(e^2)$ are retained, e being the coupling constant. Expanding the \mathcal{T} -matrix up to second order in the coupling constant, we obtain:

$$\mathcal{T}^2 - \mathcal{T}^{2\dagger} = i\mathcal{T}^1\mathcal{T}^{1\dagger}, \quad (5)$$

using the unitarity of the S -matrix. Here the superscripts on \mathcal{T} denote the order of the coupling constant in the expansion. Taking the expectation value of eq.(5) between two in-states we have:

$$\sum_{out} | \langle out | \mathcal{T}^1 | in \rangle |^2 = 2Im \langle in | \mathcal{T}^2 | in \rangle. \quad (6)$$

Depending on the choice of the initial states, we obtain the decay/production amplitude of the particles.

In order to find these rates in a thermal medium the thermal expectation value of eq.(6) has to be evaluated. There are a number of ways to perform the thermal averaging. Amongst them the more commonly used formalisms are the imaginary time technique of Matsubara [5] and the real time finite temperature formalism (see [6–10,12,13] and references therein). In the present work we use the real time formalism of the finite temperature field theory. The propagator acquires a matrix structure in this formalism and the off-diagonal elements provide the decay/production amplitudes. Unfortunately, the direct evaluation of the off-diagonal elements in presence of an external field is rather complicated. Therefore, for the ease of calculation, we work with the 11-component of the propagator and find imaginary part of the 11-component of the photon polarisation tensor ($\Pi_{\mu\nu}^{11}$). This quantity, multiplied by appropriate factors then gives the correct value for the imaginary part of the polarisation tensor [9,10,12,13]. Though for notational brevity we shall suppress the 11-superscript for both the propagator and the polarisation tensor in the rest of the paper.

At the 1-loop level, the vacuum polarization tensor arises from the diagram in fig. 1. The dominant contribution to the vacuum polarization comes from the electron line in the loop. To evaluate this diagram we use the electron propagator within a thermal medium in presence of a background electro-magnetic field. Rather than working

with a completely general background field we specialize to the case of a purely magnetic field. Once this is assumed, the field can be taken in the z -direction without any further loss of generality. We denote the magnitude of this field by \mathcal{B} . Ignoring at first the presence of the medium, the electron propagator in such a field can be written down following Schwinger's approach [15–17]:

$$\begin{aligned} iS_B^V(p) &= \int_0^\infty ds \frac{e^{ie\mathcal{B}s\sigma_z}}{\cos(e\mathcal{B}s)} \\ &\times \exp \left[is \left(p_\parallel^2 - \frac{\tan(e\mathcal{B}s)}{e\mathcal{B}s} p_\perp^2 - m^2 + i\epsilon \right) \right] \\ &\times \left(\not{p}_\parallel - \frac{e^{-ie\mathcal{B}s\sigma_z}}{\cos(e\mathcal{B}s)} \not{p}_\perp + m \right), \end{aligned} \quad (7)$$

where

$$\not{p}_\parallel = \gamma_0 p_0 - \gamma_3 p_3 \quad (8)$$

$$p_\parallel^2 = p_0^2 - p_3^2 \quad (9)$$

$$\not{p}_\perp = \gamma_1 p_1 + \gamma_2 p_2 \quad (10)$$

$$p_\perp^2 = p_1^2 + p_2^2, \quad (11)$$

and σ_z is given by:

$$\sigma_z = i\gamma_1\gamma_2 = -\gamma_0\gamma_3\gamma_5, \quad (12)$$

where the two forms are equivalent because of the definition of γ_5 . Since

$$e^{ie\mathcal{B}s\sigma_z} = \cos e\mathcal{B}s + i\sigma_z \sin e\mathcal{B}s, \quad (13)$$

we can rewrite the propagator in the following form:

$$iS_B^V(p) = \int_0^\infty ds e^{\Phi(p,s)} C(p,s), \quad (14)$$

where we have used the shorthands,

$$\Phi(p,s) \equiv is \left(p_\parallel^2 - \frac{\tan(e\mathcal{B}s)}{e\mathcal{B}s} p_\perp^2 - m^2 \right) - \epsilon|s|, \quad (15)$$

$$C(p,s) \equiv \left[(1 + i\sigma_z \tan e\mathcal{B}s)(\not{p}_\parallel + m) - (\sec^2 e\mathcal{B}s)\not{p}_\perp \right]. \quad (16)$$

Of course in the range of integration indicated in eq.(14) s is never negative and hence $|s|$ equals s . It should be mentioned here that we follow the notation adopted in paper-I to ensure continuity. In the presence of a background medium, the above propagator is modified to [18]:

$$iS(p) = iS_B^V(p) - \eta_F(p) \left[iS_B^V(p) - i\overline{S}_B^V(p) \right], \quad (17)$$

where

$$\overline{S}_B^V(p) \equiv \gamma_0 S_B^{V\dagger}(p) \gamma_0, \quad (18)$$

for a fermion propagator and $\eta_F(p)$ contains the distribution function for the fermions and the anti-fermions:

$$\begin{aligned}\eta_F(p) &= \Theta(p \cdot u) f_F(p, \mu, \beta) \\ &+ \Theta(-p \cdot u) f_F(-p, -\mu, \beta).\end{aligned}\quad (19)$$

Here, f_F denotes the Fermi-Dirac distribution function:

$$f_F(p, \mu, \beta) = \frac{1}{e^{\beta(p \cdot u - \mu)} + 1}, \quad (20)$$

and Θ is the step function given by:

$$\begin{aligned}\Theta(x) &= 1, \text{ for } x > 0, \\ &= 0, \text{ for } x < 0.\end{aligned}$$

Rewriting eq.(17) in the following form:

$$\begin{aligned}iS(p) &= \frac{i}{2} \left[S_B^V(p) + \overline{S}_B^V(p) \right] \\ &+ i(1/2 - \eta_F(p)) \left[S_B^V(p) - \overline{S}_B^V(p) \right], \\ &= iS_{\text{re}} + iS_{\text{im}}\end{aligned}\quad (21)$$

we recognise:

$$S_{\text{re}} = \frac{1}{2} \left[S_B^V(p) + \overline{S}_B^V(p) \right], \quad (22)$$

$$S_{\text{im}} = (1/2 - \eta_F(p)) \left[S_B^V(p) - \overline{S}_B^V(p) \right]; \quad (23)$$

where the subscripts *re* and *im* refer to the real and imaginary parts of the propagator. Using the form of $S_B^V(p)$ in eq.(14) we obtain the imaginary part to be:

$$\begin{aligned}iS_{\text{im}} &= (1/2 - \eta_F(p)) \left[iS_B^V(p) - i\overline{S}_B^V(p) \right], \\ &= (1/2 - \eta_F(p)) \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} C(p, s).\end{aligned}\quad (24)$$

with $\Phi(p, s)$ and $C(p, s)$ defined by eq.s.(15) and (16).

III. CALCULATION OF THE 1-LOOP VACUUM POLARIZATION

A. Identifying the Relevant Terms

The amplitude of the 1-loop diagram of fig. 1 can be written as:

$$i\Pi_{\mu\nu}(k) = - \int \frac{d^4 p}{(2\pi)^4} (ie)^2 \text{tr} [\gamma_\mu iS(p) \gamma_\nu iS(p')], \quad (25)$$

where, for the sake of notational simplicity, we have used

$$p' = p + k. \quad (26)$$

The minus sign on the right side is for a closed fermion loop and $S(p)$ is the propagator given by eq.(17). This implies:

$$\Pi_{\mu\nu}(k) = -ie^2 \int \frac{d^4 p}{(2\pi)^4} \text{tr} [\gamma_\mu iS(p) \gamma_\nu iS(p')]. \quad (27)$$

Using eq.(21) we have:

$$\begin{aligned}\Pi_{\mu\nu}(k) &= -ie^2 \int \frac{d^4 p}{(2\pi)^4} \\ &\times \text{tr} [\gamma_\mu (iS_{\text{re}}(p) + iS_{\text{im}}(p)) \\ &\times \gamma_\nu (iS_{\text{re}}(p') + iS_{\text{im}}(p'))].\end{aligned}\quad (28)$$

Then the absorptive part of the polarisation tensor is given by:

$$\begin{aligned}\Pi_{\mu\nu}^{11}(k) &= -ie^2 \int \frac{d^4 p}{(2\pi)^4} \text{tr} [\gamma_\mu iS_{\text{im}}(p) \gamma_\nu iS_{\text{im}}(p')] \quad (29) \\ &= -ie^2 \int \frac{d^4 p}{(2\pi)^4} (1/2 - \eta_F(p)) (1/2 - \eta_F(p')) \\ &\times \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_{-\infty}^{\infty} ds' e^{\Phi(p',s')} \\ &\times \text{tr} [\gamma_\mu C(p, s) \gamma_\nu C(p', s')] \\ &= -\frac{ie^2}{4} \int \frac{d^4 p}{(2\pi)^4} X(\beta, k, p, u) \\ &\times \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_{-\infty}^{\infty} ds' e^{\Phi(p',s')} \\ &\times \text{tr} [\gamma_\mu C(p, s) \gamma_\nu C(p', s')]\end{aligned}\quad (30)$$

where we have defined:

$$X(\beta, k, p) = (1 - 2\eta_F(p)) (1 - 2\eta_F(p')). \quad (31)$$

B. Extracting the Gauge Invariant Piece

As discussed earlier we are interested only in terms up-to $\mathcal{O}(\mathcal{B})$ and therefore shall drop all terms of higher order from the subsequent calculations. Now, notice that the phase factors appearing in eq.(30) are even in \mathcal{B} . Thus, to keep terms up to $\mathcal{O}(\mathcal{B})$ we need consider only the odd terms from the traces. Performing the traces the odd terms come out to be:

$$\begin{aligned}\Pi_{\mu\nu}^{11}(k, \beta) &= -ie^2 \int \frac{d^4 p}{(2\pi)^4} X(\beta, k, p) \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \\ &\times \int_{-\infty}^{\infty} ds' e^{\Phi(p',s')} R_{\mu\nu};\end{aligned}\quad (32)$$

where

$$\begin{aligned}R_{\mu\nu} &= \varepsilon_{\mu\nu 03} m^2 (\tan e\mathcal{B}s - \tan e\mathcal{B}s') \\ &+ \varepsilon_{\mu\nu \alpha\parallel\beta\parallel} \left(p^{\tilde{\alpha}\parallel} p'^{\beta\parallel} \tan e\mathcal{B}s - p'^{\tilde{\alpha}\parallel} p^{\beta\parallel} \tan e\mathcal{B}s' \right) \\ &+ \varepsilon_{\mu\nu \alpha\parallel\beta\perp} \left(p^{\tilde{\alpha}\parallel} p'^{\beta\perp} \tan e\mathcal{B}s \sec^2 e\mathcal{B}s' \right. \\ &\left. - p'^{\tilde{\alpha}\parallel} p^{\beta\perp} \tan e\mathcal{B}s' \sec^2 e\mathcal{B}s \right).\end{aligned}\quad (33)$$

In writing this expression, we have used the notation $p^{\tilde{\alpha}_{\parallel}}$, for example. This signifies a component of p which can take only the ‘parallel’ indices, i.e., 0 and 3, and is moreover different from the index α appearing elsewhere in the expression. Using now the definition of p' from eq. (26), we can write,

$$R_{\mu\nu} = R_{\mu\nu}^{(1)} + R_{\mu\nu}^{(2)}, \quad (34)$$

where,

$$R_{\mu\nu}^{(1)} = \varepsilon_{\mu\nu\alpha\parallel\beta} \left[p^{\tilde{\alpha}_{\parallel}} \tan e\mathcal{B}s + p'^{\tilde{\alpha}_{\parallel}} \tan e\mathcal{B}s' \right] k^{\beta} \quad (35)$$

and,

$$\begin{aligned} R_{\mu\nu}^{(2)} = & \tan e\mathcal{B}s \left[m^2 \varepsilon_{\mu\nu 03} + \varepsilon_{\mu\nu\alpha\parallel\beta\parallel} p^{\tilde{\alpha}_{\parallel}} p^{\beta_{\parallel}} \right. \\ & \left. + \varepsilon_{\mu\nu\alpha\parallel\beta\perp} \left(p^{\tilde{\alpha}_{\parallel}} p^{\beta_{\perp}} + p^{\tilde{\alpha}_{\parallel}} p'^{\beta_{\perp}} \tan^2 e\mathcal{B}s' \right) \right] \\ & - \tan e\mathcal{B}s' \left[m^2 \varepsilon_{\mu\nu 03} + \varepsilon_{\mu\nu\alpha\parallel\beta\parallel} p'^{\tilde{\alpha}_{\parallel}} p'^{\beta_{\parallel}} \right. \\ & \left. + \varepsilon_{\mu\nu\alpha\parallel\beta\perp} \left(p'^{\tilde{\alpha}_{\parallel}} p'^{\beta_{\perp}} + p'^{\tilde{\alpha}_{\parallel}} p^{\beta_{\perp}} \tan^2 e\mathcal{B}s' \right) \right]. \quad (36) \end{aligned}$$

Obviously, $R_{\mu\nu}^{(1)}$ is gauge invariant, i.e., $k^{\mu} R_{\mu\nu}^{(1)} = k^{\nu} R_{\mu\nu}^{(1)} = 0$. To simplify the other term, we first note that the combinations in which the parallel components of p and p' appear in eq.(36) can be simplified by using the following identity:

$$\varepsilon_{\mu\nu\alpha\parallel\beta\parallel} a^{\tilde{\alpha}_{\parallel}} b^{\beta_{\parallel}} = -\varepsilon_{\mu\nu 03} a \cdot b_{\parallel}, \quad (37)$$

which holds for any two vectors a and b . Moreover, we perform the calculations in the rest frame of the medium where $p \cdot u = p_0$. Thus the distribution function does not depend on the spatial components of p . In the last term of each square bracket of eq.(36), the integral over the transverse components of p has the following generic structure:

$$\int d^2 p_{\perp} e^{\Phi(p,s)} e^{\Phi(p',s')} \times (p^{\beta_{\perp}} \text{ or } p'^{\beta_{\perp}}). \quad (38)$$

Notice that,

$$\begin{aligned} \frac{\partial}{\partial p_{\beta_{\perp}}} \left[e^{\Phi(p,s)} e^{\Phi(p',s')} \right] &= \left(\tan e\mathcal{B}s p^{\beta_{\perp}} + \tan e\mathcal{B}s' p'^{\beta_{\perp}} \right) \\ &\times \frac{2i}{e\mathcal{B}} e^{\Phi(p,s)} e^{\Phi(p',s')}. \quad (39) \end{aligned}$$

However, this expression, being a total derivative, should integrate to zero. Thus we obtain that,

$$\tan e\mathcal{B}s p^{\beta_{\perp}} \stackrel{\circ}{=} -\tan e\mathcal{B}s' p'^{\beta_{\perp}}, \quad (40)$$

where the sign ‘ $\stackrel{\circ}{=}$ ’ means that the expressions on both sides of it, though not necessarily equal algebraically, yield the same integral. This gives,

$$\begin{aligned} p^{\beta_{\perp}} &\stackrel{\circ}{=} -\frac{\tan e\mathcal{B}s'}{\tan e\mathcal{B}s + \tan e\mathcal{B}s'} k^{\beta_{\perp}}, \\ p'^{\beta_{\perp}} &\stackrel{\circ}{=} \frac{\tan e\mathcal{B}s}{\tan e\mathcal{B}s + \tan e\mathcal{B}s'} k^{\beta_{\perp}}. \quad (41) \end{aligned}$$

Using these identities we rewrite eq. (36) in the following form:

$$R_{\mu\nu}^{(2)} = R_{\mu\nu}^{(2a)} + \varepsilon_{\mu\nu 03} R^{(2b)}, \quad (42)$$

where,

$$R_{\mu\nu}^{(2a)} = -\varepsilon_{\mu\nu\alpha\parallel\beta} \frac{\tan e\mathcal{B}s \tan e\mathcal{B}s'}{\tan e\mathcal{B}(s+s')} (p+p')^{\tilde{\alpha}_{\parallel}} k^{\beta}, \quad (43)$$

$$\begin{aligned} R^{(2b)} = & (m^2 - p_{\parallel}^2) \tan e\mathcal{B}s - (m^2 - p'^2_{\parallel}) \tan e\mathcal{B}s' \\ & - \frac{\tan e\mathcal{B}s \tan e\mathcal{B}s'}{\tan e\mathcal{B}(s+s')} (p+p') \cdot k_{\parallel}. \quad (44) \end{aligned}$$

The term called $R^{(2b)}$ does not vanish on contraction with arbitrary k^{μ} . This term is not gauge invariant, and therefore must vanish on integration. It can be shown, following the arguments in paper-I, that this term indeed vanishes. Therefore the contribution to the absorptive part of the vacuum polarization tensor which is odd in \mathcal{B} is given by:

$$\begin{aligned} \Pi_{\mu\nu}^{11}(k, \beta) = & -ie^2 \int \frac{d^4 p}{(2\pi)^4} X(\beta, k, p) \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \\ & \times \int_{-\infty}^{\infty} ds' e^{\Phi(p',s')} \left[R_{\mu\nu}^{(1)} + R_{\mu\nu}^{(2a)} \right] \\ = & -ie^2 \varepsilon_{\mu\nu\alpha\parallel\beta} k^{\beta} \int \frac{d^4 p}{(2\pi)^4} X(\beta, k, p) \\ & \times \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_{-\infty}^{\infty} ds' e^{\Phi(p',s')} \\ & \times \left[p^{\tilde{\alpha}_{\parallel}} \tan e\mathcal{B}s + p'^{\tilde{\alpha}_{\parallel}} \tan e\mathcal{B}s' \right. \\ & \left. - \frac{\tan e\mathcal{B}s \tan e\mathcal{B}s'}{\tan e\mathcal{B}(s+s')} (p+p')^{\tilde{\alpha}_{\parallel}} \right]. \quad (45) \end{aligned}$$

C. Terms Linear in External Field

Eq.(45) is one of the important results obtained in this paper. This part of the polarisation tensor is odd in powers of the external field and is purely matter dominated. Retaining terms only up-to $\mathcal{O}(\mathcal{B})$ in eq.(45) we arrive at:

$$\Pi_{\mu\nu}^{11}(k, \beta) = -ie^3 \mathcal{B} \varepsilon_{\mu\nu\alpha\parallel\beta} k^{\beta} \int \frac{d^4 p}{(2\pi)^4} X(\beta, k, p)$$

$$\begin{aligned} & \times \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_{-\infty}^{\infty} ds' e^{\Phi(p',s')} \\ & \times \left[p^{\alpha_{\parallel}} \frac{s^2}{s+s'} + p'^{\alpha_{\parallel}} \frac{s'^2}{s+s'} \right]. \end{aligned} \quad (46)$$

Making the transformation $p \rightarrow -p - k$ in the first piece we obtain:

$$\begin{aligned} \Pi_{\mu\nu}^{11}(k, \beta) &= -e^3 \mathcal{B} \varepsilon_{\mu\nu\alpha_{\parallel}\beta} k^{\beta} \int \frac{d^4 p}{(2\pi)^4} Y(\beta, k, p_0) \\ & \times \partial_{k^{\alpha_{\parallel}}} I(p, p'), \end{aligned} \quad (47)$$

where,

$$I(p, p') = \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_{-\infty}^{\infty} ds' e^{\Phi(p',s')} \frac{s'}{s+s'}, \quad (48)$$

and

$$Y(\beta, k, p_0) = X(\beta, k, p) - X(\beta, k, -p'). \quad (49)$$

Notice that p appears in $X(\beta, k, p)$ only as a combination $p.u$. Therefore, only p_0 is present in X for a medium in its rest frame. We use this fact to define $Y(\beta, k, p_0)$. From the form of $Y(\beta, k, p_0)$ it is evident that $\Pi_{\mu\nu}^{11}(k, \beta)$ to $\mathcal{O}(\mathcal{B})$ vanishes in absence of a medium. Since the behaviour of the dispersive part of the polarisation tensor is similar [4] we conclude that the term linearly dependent on the external magnetic field does not contribute to the total polarisation tensor in absence of a medium.

It is shown in the Appendix that $I(p, p')$ can be reduced to the following form:

$$I(p, p') = 2\pi^2 \delta(p^2 - m^2) \delta(p'^2 - m^2). \quad (50)$$

Therefore, we finally have:

$$\begin{aligned} \Pi_{\mu\nu}^{11}(k, \beta) &= -2\pi^2 e^3 \mathcal{B} \varepsilon_{\mu\nu\alpha_{\parallel}\beta} k^{\beta} \int \frac{d^4 p}{(2\pi)^4} Y(\beta, k, p_0) \\ & \times \delta(p^2 - m^2) \partial_{k^{\alpha_{\parallel}}} \delta(p'^2 - m^2). \end{aligned} \quad (51)$$

The delta-functions can be written in the following form:

$$\begin{aligned} \delta(p^2 - m^2) &= \frac{1}{2E_p} [\delta(p_0 - E_p) + \delta(p_0 + E_p)], \\ \delta(p'^2 - m^2) &= \delta(p_0^2 + k_0^2 + 2p_0 k_0 - E_{p'}^2), \end{aligned} \quad (52)$$

with the definitions,

$$\begin{aligned} E_p^2 &= m^2 + P^2, \\ E_{p'}^2 &= m^2 + P^2 + K^2 + 2PK \cos \theta, \end{aligned} \quad (53)$$

where P and K are the magnitudes of the spatial parts of p and k , θ being the angle between \vec{P} and \vec{K} . Using eq.s(52) to perform the dp_0 and $d\phi$ integral we obtain:

$$\begin{aligned} \Pi_{\mu\nu}^{11}(k, \beta) &= -\pi^2 e^3 \mathcal{B} \varepsilon_{\mu\nu\alpha_{\parallel}\beta} k^{\beta} \int \frac{P^2 dP d(\cos \theta)}{(2\pi)^2} \frac{1}{E_p} \\ & \times \left[Y(\beta, k, E_p) \partial_{k^{\alpha_{\parallel}}} \delta(k^2 + 2k_0 E_p - 2PK \cos \theta) \right. \\ & \left. + (E_p \rightarrow -E_p) \right] \\ &= -\frac{\pi^2}{2} e^3 \mathcal{B} \varepsilon_{\mu\nu\alpha_{\parallel}\beta} k^{\beta} \partial_{k^{\alpha_{\parallel}}} \frac{1}{K} \int \frac{PdP d(\cos \theta)}{(2\pi)^2} \frac{1}{E_p} \\ & \times \left[\delta(\cos \theta - \frac{k^2 + 2k_0 E_p}{2PK}) Y(\beta, k, E_p) \right. \\ & \left. + (E_p \rightarrow -E_p) \right] \\ &+ \frac{\pi^2}{2} e^3 \mathcal{B} \varepsilon_{\mu\nu\alpha_{\parallel}\beta} \frac{k^{\beta}}{K} \int \frac{PdP d(\cos \theta)}{(2\pi)^2} \frac{1}{E_p} \\ & \times \left[\delta(\cos \theta - \frac{k^2 + 2k_0 E_p}{2PK}) \partial_{k^{\alpha_{\parallel}}} Y(\beta, k, E_p) \right. \\ & \left. + (E_p \rightarrow -E_p) \right]. \end{aligned} \quad (54)$$

Now performing the θ integration we arrive at:

$$\begin{aligned} \Pi_{\mu\nu}^{11}(k, \beta) &= -\frac{\pi^2}{2} e^3 \mathcal{B} \varepsilon_{\mu\nu\alpha_{\parallel}\beta} k^{\beta} \partial_{k^{\alpha_{\parallel}}} \frac{1}{K} \int \frac{PdP}{2\pi} \frac{1}{E_p} \\ & \times \{Y(\beta, k, E_p) + Y(\beta, k, -E_p)\} \\ &+ \frac{\pi^2}{2} e^3 \mathcal{B} \varepsilon_{\mu\nu\alpha_{\parallel}\beta} \frac{k^{\beta}}{K} \int \frac{PdP}{2\pi} \frac{1}{E_p} \\ & \times \partial_{k^{\alpha_{\parallel}}} \{Y(\beta, k, E_p) + Y(\beta, k, -E_p)\}. \end{aligned} \quad (55)$$

It should be noted that eq.(55) is valid only if the following conditions are met with. Since, inside the delta function $-1 \leq \cos \theta \leq 1$ as $0 \leq \theta \leq \pi$, the range of integration over P must contain the interval $[P_{\min}, P_{\max}]$, where,

$$P_{\min} = -\frac{K}{2} + \frac{k_0}{2} (1 - \frac{4m^2}{k^2})^{1/2} \quad (57)$$

$$P_{\max} = \frac{K}{2} + \frac{k_0}{2} (1 - \frac{4m^2}{k^2})^{1/2}. \quad (58)$$

Since P is real the condition $k^2 \geq 4m^2$ must be satisfied. *This is an important kinematic constraint which ensures the conservation of energy-momentum in the weak field limit.* In the next sections we shall discuss a few specific background media to find the modification of the absorptive part of the polarisation tensor in presence of a background magnetic field.

It has long been realised that the amplitudes computed using the imaginary-time formalism, when continued analytically to real frequencies, usually differ from those computed using the 11-component of the real-time technique [10]. In order to make the results consistent with each other we need to multiply our result by $\tanh \beta k_0/2$ to obtain:

$$\begin{aligned}
\Pi_{\mu\nu}^{\text{abs}}(k, \beta) = & -\frac{\pi^2}{2} e^3 \mathcal{B} \varepsilon_{\mu\nu\alpha\parallel} \beta k^\beta \tanh \frac{\beta k_0}{2} \\
& \times \partial_{k^\alpha\parallel} \frac{1}{K} \int \frac{PdP}{2\pi} \frac{1}{E_p} \\
& \times \{Y(\beta, k, E_p) + Y(\beta, k, -E_p)\} \\
& + \frac{\pi^2}{2} e^3 \mathcal{B} \varepsilon_{\mu\nu\alpha\parallel} \beta \frac{k^\beta}{K} \tanh \frac{\beta k_0}{2} \int \frac{PdP}{2\pi} \frac{1}{E_p} \\
& \times \partial_{k^\alpha\parallel} \{Y(\beta, k, E_p) + Y(\beta, k, -E_p)\} . \quad (59)
\end{aligned}$$

Now eq.(59) can be further simplified using Leibnitz theorem for differentiation of an integral given by:

$$\begin{aligned}
\frac{d}{dc} \int_{a(c)}^{b(c)} F(x, c) dx = & \int_{a(c)}^{b(c)} \frac{\partial}{\partial c} F(x, c) dx \\
& + F(b, c) \frac{db}{dc} - F(a, c) \frac{da}{dc} . \quad (60)
\end{aligned}$$

Using eq.s(57) and (58) we can reduce eq.(59) to the following form:

$$\begin{aligned}
\Pi_{\mu\nu}^{\text{abs}}(k, \beta) = & -\frac{\pi^2}{2} e^3 \mathcal{B} \varepsilon_{\mu\nu\alpha\parallel} \beta k^\beta \tanh \frac{\beta k_0}{2} \left(\partial_{k^\alpha\parallel} \frac{1}{K} \right) \\
& \times \int_{P_{\min}}^{P_{\max}} \frac{PdP}{2\pi} \frac{1}{E_p} \{Y(\beta, k, E_p) + Y(\beta, k, -E_p)\} \\
& - \frac{\pi^2}{2} e^3 \mathcal{B} \varepsilon_{\mu\nu\alpha\parallel} \beta \frac{k^\beta}{K} \tanh \frac{\beta k_0}{2} \\
& \times \left[\frac{P}{E_p} \{Y(\beta, k, E_p) + Y(\beta, k, -E_p)\} \right]_{P_{\max}} \partial_{k^\alpha\parallel} P_{\max} \\
& + \frac{\pi^2}{2} e^3 \mathcal{B} \varepsilon_{\mu\nu\alpha\parallel} \beta \frac{k^\beta}{K} \tanh \frac{\beta k_0}{2} \\
& \times \left[\frac{P}{E_p} \{Y(\beta, k, E_p) + Y(\beta, k, -E_p)\} \right]_{P_{\min}} \partial_{k^\alpha\parallel} P_{\min} . \quad (61)
\end{aligned}$$

One should note the difference in sign between the different terms of the polarisation tensor in eq.(61). This opens up the possibility that for a given magnetic field, depending on the value of chemical potential and the external photon momentum, $\Pi_{\mu\nu}^{\text{abs}}(k, \beta)$ can be either positive or negative giving rise to damping or instability of the propagating photon.

IV. DISPERSION RELATION

In this section we recapitulate the dispersion relations for a photon traveling in a magnetized medium (see paper-I for details). The quadratic part of the Lagrangian with quantum and thermal correction is –

$$\mathcal{L} = \frac{1}{2} [-k^2 \tilde{g}_{\mu\nu} + \Pi_{\lambda\rho}(k)] A^\mu(k) A^\nu(-k) , \quad (62)$$

where $\tilde{g}_{\mu\nu} = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right)$. In momentum space the equation of motion takes the form:

$$[(-k^2 + \omega_0^2) g_{\mu\nu} + \Pi_{\mu\nu}^{\text{tot}}] A^\nu = 0 . \quad (63)$$

Here we assume the Lorenz gauge condition ($\partial_\mu A^\mu = 0$) and ω_0 is the plasma frequency defined by:

$$\omega_0^2 = 4e^2 \int \frac{d^3p}{(2\pi)^3 2E_p} \left(1 - \frac{P^2}{3E_p^2} \right) [f_+ + f_-] , \quad (64)$$

where we have introduced the notation

$$f_\pm = f_F(|p_0|, \mp \mu) . \quad (65)$$

The quantity $\Pi_{\mu\nu}^{\text{tot}}$ is the total vacuum polarisation tensor containing the dispersive as well as the absorptive part. Assuming the direction of photon propagation to be along the direction of the magnetic field, for the two transverse components of the photon field A^μ , we obtain the following dispersion relations:

$$k^2 = \omega_0^2 \pm (a_{\text{disp}} + a_{\text{abs}}) , \quad (66)$$

where a_{disp} and a_{abs} are the dispersive and the absorptive parts of Π_{12} respectively. Recall that a circularly polarised wave undergoes an exponential damping through the factor $e^{-\text{Im}k_0 t}$. Therefore, the lifetime τ of a photon is given by the imaginary part of k_0 where k_0 is given by:

$$k_0^2 = \omega_0^2 + K^2 \pm a_{\text{disp}} \mp ia_{\text{abs}} . \quad (67)$$

V. LIMITING CASES

A. Non-Relativistic Fermion–Anti-Fermion Plasma

Let us consider the case of a non-degenerate fermion–anti-fermion plasma. Using eq.(19) we can write:

$$2\eta_F(p) - 1 = \epsilon(p.u) \tanh \left(\frac{\beta(p.u - \mu)}{2} \right) , \quad (68)$$

where,

$$\begin{aligned}
\epsilon(x) = & 1, \text{ for } x > 0 , \\
& -1, \text{ for } x < 0 . \quad (69)
\end{aligned}$$

This renders the thermal factors in the polarisation tensor in the following form:

$$\begin{aligned}
(1 - 2\eta_F(p)) (1 - 2\eta_F(p')) = & \epsilon(p.u) \epsilon(p'.u) \\
& \times \left[1 - \coth \frac{\beta k.u}{2} \left(\tanh \frac{\beta(p'.u - \mu)}{2} \right. \right. \\
& \left. \left. - \tanh \frac{\beta(p.u - \mu)}{2} \right) \right] , \quad (70)
\end{aligned}$$

where we have used the following identity:

$$1 - \tanh A \tanh B = \coth(A - B) (\tanh A - \tanh B).$$

Notice that in eq.(70) only the second term within the square bracket contributes to the finite temperature effect, the first one giving the zero temperature case. Using eq.s (31), (49), (70) and using the relation $\tanh z = 1 - \frac{2}{e^{2z}+1}$ one can write,

$$Y(\beta, k, p_0) = 2\epsilon(p_0)\epsilon(p'_0)\coth\frac{\beta k_0}{2} \times \left[\frac{1}{e^{\beta(p_0+\mu)}+1} - \frac{1}{e^{\beta(p_0-\mu)}+1} + \frac{1}{e^{\beta(p'_0-\mu)}+1} - \frac{1}{e^{\beta(p'_0+\mu)}+1} \right]. \quad (71)$$

Therefore, for the thermal part, using eq. (71) we obtain,

$$Y(\beta, k, E_p) + Y(\beta, k, -E_p) = 2\epsilon(E_p)\coth\frac{\beta k_0}{2} \times \left(\epsilon(E_p + k_0) \left(\frac{1}{e^{\beta(E_p+k_0-\mu)}+1} - \frac{1}{e^{\beta(E_p+k_0+\mu)}+1} - \frac{1}{e^{\beta(E_p-\mu)}+1} + \frac{1}{e^{\beta(E_p+\mu)}+1} \right) + \epsilon(E_p - k_0) \left(\frac{1}{e^{\beta(E_p-k_0-\mu)}+1} - \frac{1}{e^{\beta(E_p-k_0+\mu)}+1} - \frac{1}{e^{\beta(E_p-\mu)}+1} + \frac{1}{e^{\beta(E_p+\mu)}+1} \right) \right) \quad (72)$$

For a non-relativistic non-degenerate plasma one can approximate E_p inside the integration by the particle mass m . For a highly energetic photon with $k_0 \gg m$ we can write:

$$Y(\beta, k, E_p) + Y(\beta, k, -E_p) = 2\coth\frac{\beta k_0}{2} \times \left(\left[\frac{1}{e^{\beta(m+k_0-\mu)}+1} - \frac{1}{e^{\beta(m+k_0+\mu)}+1} - \frac{1}{e^{\beta(m-k_0-\mu)}+1} + \frac{1}{e^{\beta(m-k_0+\mu)}+1} \right] \right) \simeq 2\coth\frac{\beta k_0}{2} \left(e^{\beta k_0} - e^{-\beta k_0} \right) \times \left(e^{-\beta(m+\mu)} - e^{-\beta(m-\mu)} \right). \quad (73)$$

Using the form of the thermal factors given by eq. (73) in eq. (61) the absorptive part of the polarisation tensor becomes:

$$\Pi_{\mu\nu}^{\text{abs}}(k, \beta) = -\pi^2 e^3 \mathcal{B} \epsilon_{\mu\nu\alpha\parallel} \beta k^\beta \left(e^{\beta k_0} - e^{-\beta k_0} \right)$$

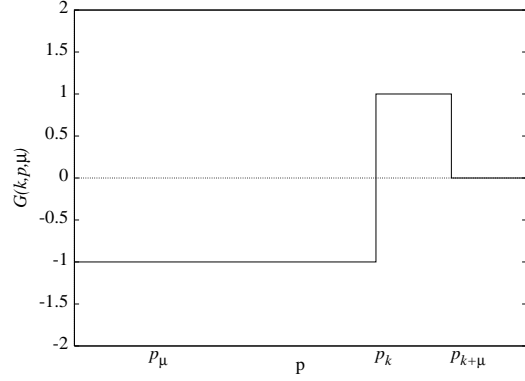


FIG. 2. $G(k, p, \mu)$ vs. p for a Fermi-degenerate plasma.

$$\times \left(e^{-\beta(m+\mu)} - e^{-\beta(m-\mu)} \right) \times \left[\left(\partial_{k^\alpha} \sim \frac{1}{K} \right) \left(\frac{K}{2\pi} \right) - \partial_{k^\alpha} \sim K \right] \quad (74)$$

From eq. (74) one can easily check that in the limit of $m \ll \mu$ the absorptive part of the polarisation tensor becomes negative signifying damping of the propagating photon.

B. Degenerate Fermion Gas

In this section we consider a degenerate fermion gas at zero temperature. For degenerate fermions eq.(72) reduces to,

$$Y(\beta, k, E_p) + Y(\beta, k, -E_p) = 2\epsilon(E_p)\coth\frac{\beta k_0}{2} \times \left(\epsilon(E_p + k_0) \left(\frac{1}{e^{\beta(E_p+k_0-\mu)}+1} - \frac{1}{e^{\beta(E_p-\mu)}+1} \right) + \epsilon(E_p - k_0) \left(\frac{1}{e^{\beta(E_p-k_0-\mu)}+1} - \frac{1}{e^{\beta(E_p-\mu)}+1} \right) \right). \quad (75)$$

Let us consider the special case of relativistic fermions (for example the relativistically degenerate electrons in white dwarfs or neutron stars) and very high energy photons such that $m^2 < k^2$ and $k_0 \pm K > 2\mu$. With this assumption we have $P_{max} = K/2 + k_0/2$ and $P_{min} = k_0/2 - K/2$. Because of the high virtuality of the photon ($k^2 \gg m^2$) we can assume that $k_0 \gg K$. It is now easy to see that for a degenerate Fermi gas at $T = 0$ the factor inside the bracket (let us denote it by $G(k, p, \mu)$ of eq.(75) is given by fig.2, where we have used the following definitions:

$$p_\mu = \sqrt{\mu^2 - m^2}, \quad (76)$$

$$p_k = \sqrt{k_0^2 - m^2}, \quad (77)$$

$$p_{k+\mu} = \sqrt{(\mu + k_0)^2 - m^2}. \quad (78)$$

It is evident that in the interval $[P_{\min}, P_{\max}]$, for our choice of the kinematics, G is entirely negative. Therefore the field dependent part of the thermal polarisation tensor takes the form:

$$\begin{aligned}\Pi_{\mu\nu}^{\text{abs}}(k, \beta) &= \frac{\pi^2}{2} e^3 \mathcal{B} \varepsilon_{\mu\nu\alpha\parallel\beta} k^\beta \left(\frac{\partial}{\partial k^\alpha} \frac{1}{K} \right) \\ &\times \int_{P_{\min}}^{P_{\max}} \frac{PdP}{2\pi} \frac{1}{E_p} \\ &+ \frac{\pi^2}{2} e^3 \mathcal{B} \varepsilon_{\mu\nu\alpha\parallel\beta} \frac{k^\beta}{K} \left[\frac{P}{E_p} \right]_{P_{\min}}^{\partial \sim P_{\max}} \frac{\partial}{\partial k^\alpha} \frac{1}{K} \\ &- \frac{\pi^2}{2} e^3 \mathcal{B} \varepsilon_{\mu\nu\alpha\parallel\beta} \frac{k^\beta}{K} \left[\frac{P}{E_p} \right]_{P_{\min}}^{\partial \sim P_{\min}} \frac{\partial}{\partial k^\alpha} \frac{1}{K} \\ &\simeq \frac{\pi^2}{2} e^3 \mathcal{B} \varepsilon_{\mu\nu\alpha\parallel\beta} k^\beta \left(\frac{\partial}{\partial k^\alpha} \frac{1}{K} \right) \\ &\times m \left[\frac{1}{1 + \left(\frac{k_0 + K}{2m} \right)^2} - \frac{1}{1 + \left(\frac{k_0 - K}{2m} \right)^2} \right] \quad (79)\end{aligned}$$

To derive the above expression we have assumed a very high energy photon i.e., $P_{\max}^2 + m^2 \sim P_{\min}^2 + m^2$ in the limit of $k_0 \gg K$. Further assuming the photon to be moving in a direction parallel to the direction of the magnetic field we obtain,

$$\Pi_{\mu\nu}^{\text{abs}}(k, \beta) \sim 4\pi^2 e^3 \mathcal{B} \varepsilon_{\mu\nu 03} \frac{m^3}{k_0^3}. \quad (80)$$

Dispersive Part of $\Pi_{\mu\nu}$ – Here we evaluate field dependent part of the real part of the vacuum polarisation tensor for a degenerate fermion gas. Our starting point is eq.(5.21) in [4]. For a weak magnetic field it can be shown that:

$$\begin{aligned}\Pi_{\mu\nu}^{\text{disp}}(k, \beta) &= \frac{\pi^2}{2} e^3 \mathcal{B} \varepsilon_{\mu\nu\alpha\parallel\beta} k^\beta \int \frac{d^4p}{(2\pi)^4} \epsilon(p_0) f_+(p) \\ &\times \int_{-\infty}^{\infty} e^{is(p^2 - m^2) - \epsilon|s|} \int_0^{\infty} e^{is'(p'^2 - m^2) - \epsilon s'} \\ &\times \left[\left(p^{\alpha\parallel}(s + s') - 2\tilde{p}^{\alpha\parallel}s + 2\tilde{p}^{\alpha\parallel} \frac{s^2}{s + s'} \right) \right. \\ &\left. + k^{\alpha\parallel} \left((s + s') - 2s + \frac{s^2}{(s + s')} \right) \right]. \quad (81)\end{aligned}$$

Because of the kinematics and the condition that the photon propagates along the z direction, the term proportional to $k^{\alpha\parallel}$ would vanish. Now the non-vanishing piece comes from the term proportional to $\tilde{p}^{\alpha\parallel}$. Using arguments as has been used previously in [4], we can write,

$$I = i \frac{\partial}{\partial(m^2)} J_0 - 2J_1 + i \int d(m^2) J_2 \quad (82)$$

With the difference that contrary to [4] here, powers of s appear in the numerator instead of s' in the second and third term. The J s are defined as,

$$\begin{aligned}J_n &= \int \frac{d^4p}{(2\pi)^4} \epsilon(p_0) f_+(p) \int_{-\infty}^{\infty} e^{is(p^2 - m^2) - \epsilon|s|} \\ &\times \int_0^{\infty} e^{is'(p'^2 - m^2) - \epsilon s'} s^n. \quad (83)\end{aligned}$$

Out of the three terms in eq.(81) the last two terms do not contribute. Therefore, we are left with only the first term. The s and s' integration in J_0 leaves us with,

$$\begin{aligned}I &= \frac{\partial}{\partial(m^2)} \int \frac{d^4p}{(2\pi)^4} \epsilon(p_0) f_+(|p_0|) \tilde{p}^{\alpha\parallel} \frac{\delta(p^2 - m^2)}{(p'^2 - m^2)}, \\ &\simeq \frac{\partial}{\partial(m^2)} \int \frac{d^3p}{(2\pi)^4} f_+(|p_0|) \\ &\times \left(\frac{1}{(k^2 + 2E_p k_0)} - \frac{1}{(k^2 - 2E_p k_0)} \right) \quad (84)\end{aligned}$$

with $E_p = \sqrt{\vec{p}^2 + m^2}$. In the last step we have neglected terms proportional to $\vec{P} \cdot \vec{k}$ in the denominator, since P_F^2 , $|K|^2$ and m^2 are negligible compared to $k^2 \sim k_0^2$. The dominant behavior of eq.(81) is,

$$I = -\frac{P_F^2}{2k_0^3}. \quad (85)$$

So the real part of the of the polarisation tensor, in a weak magnetic field and degenerate fermion gas, comes out to be,

$$\Pi_{\mu\nu}^{\text{disp}}(k, \beta) = -4ie^3 \mathcal{B} \varepsilon_{\mu\nu 30} \frac{P_F^2}{2k_0^3}. \quad (86)$$

Now we can find out the lifetime of a photon in the plasma using eq.(67). Note that for a degenerate plasma $\omega_0 \sim P_F$ and hence this is more dominant than the contribution from Π^{abs} . Therefore, we neglect Π^{abs} while estimating photon damping time. With a little bit of algebra one can show that

$$\text{Im}(k_0) \sim 4\pi^2 e^3 \mathcal{B} \frac{m^3}{\omega_0 k_0^3}, \quad (87)$$

the inverse of which is the life-time of the photon.

VI. CONCLUSION

We have calculated the absorptive part of the polarisation tensor in a magnetised medium to 1-loop order. Our most general result is odd in powers of $e\mathcal{B}$ summed to all orders. In that we have retained terms up-to the linear order in $e\mathcal{B}$ and estimated the damping rate of a photon propagating along the magnetic field. We have specialised to a degenerate fermionic system where the kinematics of the process under consideration is $k^2 > m^2$ and $k_0 \pm K \gg 2\mu$ with $T \sim 0$. In this domain we see that the imaginary part of ω has a positive sign which

signifies that the propagating photon gets damped by creation of fermion anti-fermion pairs. If we take the inverse, it would give us the time scale of damping. Since $e\mathcal{B}/m^2$ is small the damping or attenuation time would be rather large.

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APPENDIX:

Evaluation of $I(p, p')$ - Recall that $I(p, p')$ is given by,

$$\begin{aligned} I(p, p') &= \int_{-\infty}^{\infty} ds e^{\Phi(p, s)} \int_{-\infty}^{\infty} ds' e^{\Phi(p', s')} \frac{s'}{s' + s} \\ &= \frac{1}{2} \int_{-\infty}^{\infty} ds e^{\Phi(p, s)} \int_{-\infty}^{\infty} ds' e^{\Phi(p', s')} \left[1 + \frac{s' - s}{s' + s} \right] \\ &= 2\pi^2 \delta(p^2 - m^2) \delta(p'^2 - m^2) + \frac{1}{2} I'(p, p') \quad (A1) \end{aligned}$$

where we have defined :

$$\begin{aligned} I'(p, p') &= \int_{-\infty}^{\infty} ds e^{\Phi(p, s)} \int_{-\infty}^{\infty} ds' e^{\Phi(p', s')} \left[\frac{s' - s}{s' + s} \right] \\ &= \int_{-\infty}^{\infty} \frac{dt}{t} e^{it(p^2 + p \cdot k + k^2/2 - m^2 - i\epsilon_1)} \\ &\quad \times \int_{-\infty}^{\infty} dt' t' e^{it'(p \cdot k + k^2/2 - i\epsilon_2)} \quad (A2) \end{aligned}$$

where $t = s + s'$ and $t' = s' - s$. Now,

$$\begin{aligned} &\int_{-\infty}^{\infty} dt' t' e^{it'(p \cdot k + k^2/2 - i\epsilon_2)} \\ &= \frac{1}{2i\pi} \left\{ \frac{1}{(p \cdot k + k^2/2 + i\epsilon_2)^2} - \frac{1}{(p \cdot k + k^2/2 - i\epsilon_2)^2} \right\} \\ &= \frac{2(p \cdot k + k^2/2)}{(p \cdot k + k^2/2)^2 + \epsilon_2^2} \delta(p \cdot k + k^2/2) \quad (A3) \end{aligned}$$

where we have used the following identity :

$$\frac{1}{a \pm i\epsilon} = \mathcal{P}(a) \mp i\pi\delta(a), \quad \mathcal{P} \text{ being the principal value.} \quad (A4)$$

Therefore we have,

$$\begin{aligned} I'(p, p') &= \delta(p \cdot k + k^2/2) \frac{2(p \cdot k + k^2/2)}{(p \cdot k + k^2/2)^2 + (\epsilon_2)^2} \\ &\quad \times \int_{-\infty}^{\infty} \frac{dt}{t} e^{it(p^2 + p \cdot k + k^2/2 - m^2 - i\epsilon_1)} \quad (A5) \end{aligned}$$

Since the numerator and the argument of the delta function are the same, $I'(p, p')$ vanishes upon p -integration, provided we take the limit $\epsilon_2 \rightarrow 0^+$ later. Therefore, finally we are left with :

$$I(p, p') = 2\pi^2 \delta(p^2 - m^2) \delta(p'^2 - m^2). \quad (A6)$$

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